MICROCOPY

AD-A169 262

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER ARO 19367.36-MA	2. GOVT ACCESSION NO. N/A	3. RECIPIENT'S CATALOG NUMBER N/A
4. TITLE (and Subtitle) A Reversal Argument for Storage Models Defined on Semi-Markov Processes		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Eric S. Tollar		8. CONTRACT OR GRANT NUMBER(s) DAAG29-82-K-0168
9. PERFORMING ORGANIZATION NAME AND ADDRESS The Florida State University Tallahassee, FL 32306-3033		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office Post Office Box 12211 Research Triangle Park, NC 27709		12. REPORT DATE February, 1986
		13. NUMBER OF PAGES 35
14. MONITORING AGENCY NAME & ADDRESS(If different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

DTIC
ELECTE
JUN 2 7 1986
S
D
D

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, If different from Report)

NA

18. SUPPLEMENTARY NOTES

The view, opinions, and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

For many storage models defined on some semi-Markov process $X(t)$, the asymptotic distribution of the imbedded discrete time process can often be determined by exploiting the properties of the dual of the underlying

DD FORM 1473 1 JAN 73   EDITION OF 1 NOV 65 IS OBSOLETE

20. ABSTRACT CONTINUED

Markov chain X_n, which effectively reverses the process. If this is the case, a technique is given which under certain regularity conditions shows the asymptotic distribution of the entire continuous time process can be obtained, and is equal to an altered version of the "reversed" discrete time process. It is shown this method not only can be applied to models where the asymptotic distribution was previously unknown, but can also improve upon characterizing many of the results for models in which the asymptotic behavior is obtained by a renewal argument.

Accesion For		
NTIS CRA&I	☑	
DTIC TAB	☐	
Unannounced	☐	
Justification		
By		
Distribution/		
Availability Codes		
Dist	Avail and/or Special	
A-1		

A Reversal Argument for Storage
Models Defined on Semi-Markov Processes

by

Eric S. Tollar

FSU Technical Report No. M721
USARO Technical Report No. D-85

February, 1986

The Florida State University
Department of Statistics
Tallahassee, Florida 32306-3033

A Reversal Argument for Storage
Models Defined on Semi-Markov Processes

by

Eric S. Tollar

Abstract

For many storage models defined on some semi-Markov process $X(t)$, the
asymptotic distribution of the imbedded discrete time process can often be
determined by exploiting the properties of the dual of the underlying
Markov chain X_n, which effectively reverses the process. If this is the
case, a technique is given which under certain regularity conditions shows
the asymptotic distribution of the entire continuous time process can be
obtained, and is equal to an altered version of the "reversed" discrete
time process. It is shown this method not only can be applied to models
where the asymptotic distribution was previously unknown, but can also
improve upon characterizing many of the results for models in which the
asymptotic behavior is obtained by a renewal argument.

1. INTRODUCTION

In storage models, the concept of using an underlying Markov renewal process to allow for some dependency of structure, as well as continuity of time, has seen widespread usage. In such models, the investigation of the limit behavior of the contents in storage as time tends to infinity has always been one of the more important aspects of the model, and there have been a wide variety of techniques used in the literature to determine the limit behavior.

When the structure is such that the amount in storage, when coupled with the state of the underlying semi-Markov process, is itself a semi-Markov process on some arbitrary state space, the general theory of semi-Markov processes on arbitrary state spaces (for example, see Çinlar (1969), Athreya, McDonald and Ney (1978a, 1978b), Athreya and Ney (1978), Kesten (1974), and Nummelin (1978)) can be exploited. This technique was successfully exploited by Puri and Tollar (1985) to determine the limit behavior of a popular storage model.

Another popular technique is to "reverse" the process by looking at the dual Markov renewal process (for a summary of the full power of the dual process, see Kemeny, Snell and Knapp (1976)). While this method has proven itself useful in the limit behavior in those cases where the contents in storage require some normalization (see, for example, Puri and Woolford (1981)), it has failed in those cases where no normalization is required.

In this paper, under certain assumptions on the definition of the storage model, we present a technique which allows one to "reverse" the continuous time process to obtain results in those cases where no normalization is required. Because of the assumptions imposed on the model, these results can also be considered as an extension of the theory of semi-Markov processes on arbitrary state spaces (although there is admittedly more structure on our state space than the previously cited authors prefer to allow).

Let J be a subset of the integers, and $\{X_n, n = 0, 1, 2, \ldots\}$ be a stationary, irreducible, aperiodic, positive recurrent Markov chain with transition matrix $\underset{\sim}{P} = (p_{ij})$ for $i, j \in J$, and with stationary measure $\underset{\sim}{\pi}$. We then define times $0 \equiv T_0 \leq T_1 \leq T_2 \ldots$ such that $\{(X_n, T_n), n = 0, 1, 2, \ldots\}$ is a Markov renewal process with semi-Markov matrix $\underset{\sim}{A}(t) = (A_{ij}(t))$, where for $i, j \in J$, $t \geq 0$,

$$P(X_n = j, \ T_n - T_{n-1} \leq t \mid T_0, \ X_0, \ T_1, \ X_1, \ \ldots, \ X_{n-1} = i) = A_{ij}(t)$$

(see Çinlar (1975) for details). For all $i \in J$, define the expected sojourn time in state i by

$$m_i = \sum_{j \in J} \int_0^\infty t dA_{ij}(t),$$

and define the average sojourn time of the process by

$$\beta = \sum_{i \in J} \pi_i m_i. \tag{1.1}$$

For all $t \geq 0$, define the number of jumps by time t by

$$N(t) = \sup \{n : T_n \leq t\}.$$

We define another Markov renewal process $\{(\hat{X}_n, \hat{T}_n)\}$, independent of $\{(X_n, T_n)\}$, with semi-Markov matrix $\hat{\underline{A}}(t) = (\hat{A}_{ij}(t))$ defined for each $i, j \in J$ by

$$\hat{A}_{ij}(t) = \pi_j \pi_i^{-1} A_{ji}(t),$$

and let $\hat{X}_0$ have initial distribution $\underline{\pi}$.

Definition. $\{(\hat{X}_n, \hat{T}_n)\}$ as defined above is called the _dual_ Markov renewal process of $\{(X_n, T_n)\}$, and $\{\hat{X}_n\}$ is called the _dual_ Markov chain.

Finally, for each $i \in J$, we associate a sequence of i.i.d. random variables $\{U_n(i), n = 0, 1, 2, \ldots\}$ on some arbitrary state space, where $\{U_n(i)\}$ is independent of $\{(X_n, T_n)\}$ and $\{(\hat{X}_n, \hat{T}_n)\}$, and of all $\{U_n(j)\}$ for $j \neq i$. We then define the contents in storage at jump n recursively for some function f by

$$Z_n(x) = f(Z_{n-1}(x), U_n(X_n)), \qquad (1.2)$$

where $Z_0(x) \equiv x$.

For simplicity, we will assume the function $f(\cdot)$ is real-valued. To define the amount in storage at time t, we let

$$Z(t) = Z_{N(t)}(Z_0).$$

If we define a sequence of functions recursively by

$$f^{(n)}(x; y_1, y_2, \ldots, y_n) = f(f^{(n-1)}(x; y_1, y_2, \ldots, y_{n-1}), y_n)$$

we can represent $Z_n(x)$ by

$$Z_n(x) = f^{(n)}(x; U_1(X_1), U_2(X_2), \ldots, U_n(X_n)).$$

Define $\hat{Z}_n(x)$ by

$$\hat{Z}_n(x) = f^{(n)}(x; U_{n-1}(\hat{X}_{n-1}), U_{n-2}(\hat{X}_{n-2}), \ldots, U_0(\hat{X}_0)).$$

For convenience, let

$$g^{(n)}(x; i_1, i_2, \ldots, i_n) = f^{(n)}(x; U_{n-1}(i_1), (U_{n-2}(i_2), \ldots, U_0(i_n)). \quad (1.3)$$

Note from Kemeny Snell and Knapp (1976) that if X_0 has initial distribution π, $\hat{Z}_n(x) \overset{d}{=} Z_n(x)$, for all n.

We will assume throughout the paper the following condition on the storage model of (1.2): for any arbitrary distribution of X_0, for each $B > 0$, and each $\epsilon > 0$, there is an N where for all $n > N$,

$$P(\sup_{|x| \leq R} | Z_n(0) - Z_n(x) | > \epsilon) < \epsilon. \quad (1.4)$$

As such, the initial amount of the contents in storage is uniformly forgotten as n tends to infinity.

The importance of the condition and the usefulness of the reversibility in discrete time n can be seen from the following theorem. This theorem is stated without proof, for even though it is not explicitly stated in the literature, the techniques required to prove it are well established (see for example Puri and Woolford (1981)).

THEOREM 1.1.   If condition (1.4) is satisfied, then the

$\lim_{n\to\infty} P(X_n = i, Z_n(x) \le y)$ exists for all x, and for all initial distributions

of X_0, if an only if $\lim_{n\to\infty} P(\hat{X}_0 = i, \hat{Z}_n(0) \le y)$ exists, in which case for all

continuity points y of $\lim_{n\to\infty} P(\hat{X}_0 = i, \hat{Z}_n(0) \le y)$, for all x, and all $i \in J$

$$\lim_{n\to\infty} P(X_n = i, Z_n(x) \le y) = \lim_{n\to\infty} P(\hat{X}_0 = i, \hat{Z}_n(0) \le y).$$

Therefore, in discrete time, one can either examine the original process
or the dual process, whichever is more convenient.

It is easy to see that $\{X_n, Z_n(Z_0), T_n\}$ is itself a Markov renewal
process. However, the state space of $\{X_n, Z_n(Z_0)\}$ need not be denumerable.
Therefore, even if it is possible to establish that $\{X_n, Z_n\}$ converges in
distribution as n tends to infinity, it need not follow that $\{X(t), Z(t)\}$
converges in distribution. The substantial body of work on semi-Markov
processes on arbitrary state spaces cited previously is of little help
in proving what appears should be true: as long as $\beta < \infty$ in (1.1), then
(X_n, Z_n) converging should imply $\{X(t), Z(t)\}$ converges.

It will be shown in section 2 that under certain conditions a re-
versibility argument can be applied to determine the convergence in
distribution as time tends to infinity. Section 3 is then devoted to
applications of the results in section 2 to some examples of storage
models.

2. THE REVERSIBILITY ARGUMENT

If β of (1.1) is finite, in addition to and independent of all the random variables defined in section 1, let $\{X_n^*, n = 0, 1, \ldots\}$ be a Markov chain with state space J, transition probability for $i, j \in J$ of

$$P_{ij}^* = P(X_n^* = j \mid X_{n-1}^* = i) = \pi_j \pi_i^{-1} r_{ji},$$

and initial distribution of X_0^* given by

$$P(X_0^* = i) = \beta^{-1} \pi_i m_i.$$

We also define $Z_n^*(x)$ by

$$Z_n^*(x) = f^{(n)}(x; U_{n-1}(X_{n-1}^*), \ldots, U_0(X_0^*)) = g^{(n)}(x; X_{n-1}^*, X_{n-2}^*, \ldots, X_0^*)$$

as defined in (1.3). The relation between $Z_n^*(x)$ and $Z_n(x)$ is given in the theorem below.

THEOREM 2.1.   For each $r > 0$ and each $\epsilon > 0$, there is an N where for all $n > N$,

$$P(\sup_{|x| \leq r} \mid Z_n^*(0) - Z_n^*(x) \mid > \epsilon) < \epsilon.$$

In addition, if $(X_n, Z_n(0))$ converges in distribution, then $(X_n^*, Z_n^*(0))$ converges in distribution.

PROOF.   Let X_0 have initial distribution π, and let K be a finite subset of J, where both

$$\sum_{i \in K} \pi_i > 1 - \epsilon \quad \text{and} \quad \sum_{i \in K} \beta^{-1} \pi_i m_i > 1 - \epsilon.$$

From (1.4), for $B > 0$, $\epsilon > 0$, we can select an N where for $n > N$,

$$P(\sup_{|x| \leq B} |Z_n(0) - Z_n(x)| > \epsilon) < \epsilon \min_{i \in K}(\pi_i).$$

Then $P(\sup_{|x| \leq B} |Z_n^*(0) - Z_n^*(x)| > \epsilon) = \sum_{i \in J} P(\sup_{|x| \leq B} |Z_n^*(0) - Z_n^*(x)| > \epsilon | X_0^* = i) \beta^{-1} \pi_i m_i$

$$= \sum_{i \in J} P(\sup_{|x| \leq B} |\hat{Z}_n(0) - \hat{Z}_n(x)| > \epsilon | \hat{X}_0 = i) \beta^{-1} \pi_i m_i$$

$$= \sum_{i \in J} \pi_i^{-1} P(\sup_{|x| \leq B} |\hat{Z}_n(0) - \hat{Z}_n(x)| > \epsilon, \hat{X}_0 = i) \beta^{-1} \pi_i m_i. \qquad (2.1)$$

From Kemeny, Snell and Knapp (1976), we have for $i \in K$,

$$P(\sup_{|x| \leq B} |\hat{Z}_n(0) - \hat{Z}_n(x)| > \epsilon, \hat{X}_0 = i)$$

$$= P(\sup_{|x| \leq B} |Z_n(0) - Z_n(x)| > \epsilon, X_n = i) < \epsilon \min_{i \in K}(\pi_i).$$

Thus from (2.1) and the definition of K it follows that

$$P(\sup_{|x| \leq B} |Z_n^*(0) - Z_n^*(x)| > \epsilon) < \sum_{i \in K} \pi_i^{-1} \epsilon \min_{i \in K}(\pi_i) \cdot \beta^{-1} \pi_i m_i + \epsilon < 2\epsilon.$$

Also, again letting X_0 have distribution η, we have from theorem 1.1 that $(X_n, Z_n(0))$ converging in distribution implies for all continuity points of y,

$$\lim_{n \to \infty} P(\hat{X}_0 = i, \hat{Z}_n(0) \leq y) = P(\hat{X}_0 = i, \hat{Z} \leq y), \text{ for some } \hat{Z}.$$

Therefore,

$$\lim_{n \to \infty} P(X_0^* = i, Z_n^*(0) \leq y) = \lim_{n \to \infty} P(Z_n^*(0) \leq y \mid X_0^* = i)\beta^{-1}\pi_i m_i$$

$$= \lim_{n \to \infty} P(\hat{Z}_n(0) \leq y \mid \hat{X}_0 = i)\beta^{-1}\pi_i m_i = P(\hat{X}_0 = i, \hat{Z} \leq y)\beta^{-1}m_i.$$

As such, we have shown that $(X_0^*, Z_n^*(0))$ converges in distribution. ☐

The fundamental lemma of this paper makes clear the relation between $\{X_n^*\}$ and the behavior of the continuous time semi-Markov process.

LEMMA 2.2.   If $\beta < \infty$, then for any $k > 0$

$$\lim_{t \to \infty} P(X_{N(t)} = i_0, X_{N(t) - 1} = i_1, \ldots, X_{N(t) - k} = i_k)$$

$$= P(X_0^* = i_0, X_1^* = i_1, \ldots, X_k^* = i_k).$$

PROOF.   For a particular k, let us define a new Markov renewal process $\{(Y_n, T_n), n = 0, 1, \ldots\}$ with state space

$$K = \{(i_0, i_1, \ldots, i_k) \in J^{k+1}: p_{i_0 i_1} \cdot p_{i_1 i_2} \cdots p_{i_{k-1} i_k} > 0\}$$

by

$$P(Y_n = (j_0, j_1, \ldots, j_k), T_n - T_{n-1} \le t \mid Y_{n-1} = (i_0, i_1, \ldots, i_k))$$

$$= A_{i_k j_k}(t) \prod_{\ell=1}^{k} I(i_\ell = j_{\ell - 1}).$$

As is apparent, $\{(Y_n, T_n)\}$ is merely the process $\{(X_n, T_n)\}$ with memory of the previous k states it has visited. Therefore, for $N(t) > k$,

$$P(X_{N(t) - k} = i_k, X_{N(t) - k + 1} = i_{k-1}, \ldots, X_{N(t)} = i_0)$$

$$= P(Y_{N(t)} = (i_k, i_{k-1}, \ldots, i_0)).$$

Also, $\{Y_n\}$ must be aperiodic and irreducible, since $\{X_n\}$ is. Finally, letting

$$\pi^*(j_0, j_1, \ldots, j_k) = \pi_{j_0} p_{j_0 j_1} \cdots p_{j_{k-1} j_k},$$

we have that

$$\sum_{(i_0, i_1, \ldots, i_k) \in K} \pi^*(i_0, i_1, \ldots, i_k)$$

$$\cdot P(Y_n = (j_0, \ldots, j_k) \mid Y_{n-1} = (i_0, i_1, \ldots, i_k))$$

$$= \sum_{i_0 \in \mathcal{I}} (\pi_{i_0} p_{i_0 j_0} p_{j_0 j_1} \cdots p_{j_{k-2} j_{k-1}})^{\sim} p_{j_{k-1} j_k} = \pi^*(j_0, j_1, \ldots, j_k).$$

As such, we have from Karlin and Taylor (1975) that Y_n is positive recurrent with stationary measure $\underset{\sim}{\pi}^*$. Therefore, from Cinlar (1975), it follows that

$$\lim_{t \to \infty} P(X_{N(t) - 1} = i_k, \; X_{N(t) - k + 1} = i_{k-1}, \; \ldots, \; X_{N(t)} = i_0)$$

$$= \frac{\pi_{i_k} p_{i_k i_{k-1}} \cdots p_{i_1 i_0} p_{i_0}}{\sum\limits_{(i_k, \, i_{k-1}, \, \ldots \, i_0) \in K} \pi_{i_k} p_{i_k i_{k-1}} \cdots p_{i_1 i_0} \pi_{i_0}}$$

$$= \beta^{-1} \pi_{i_k} p_{i_k i_{k-1}} \cdots p_{i_1 i_0} p_{i_0}.$$

The proof is completed once it is observed that

$$P(X_0^* = i_0, \; \ldots, \; X_k^* = i_k) = \beta^{-1} \pi_{i_k} p_{i_k i_{k-1}} \cdots p_{i_1 i_0} p_{i_0}. \quad \square$$

Thus we see that in continuous time, the Markov renewal process can in a sense be "reversed" to look like the dual Markov chain with a different initial distribution.

From this lemma, we can establish the main theorem of this paper.

THEOREM 2.3. If $\beta < \infty$, if $(Y_n, Z_n(z_0)) \xrightarrow{d} (X, Z)$, and if for all $\epsilon > 0$, there is a B where for all $k > 0$ $\overline{\lim\limits_{t \to \infty}} P(|Z_{N(t) - k}| > B) < \epsilon$, then

$$\lim_{t \to \infty} P(X(t) = i, \; Z(t) \le y) = \lim_{n \to \infty} P(X_0^* = i, \; Z_n^*(0) \le y), \text{ for all continuity points } y.$$

PROOF. Let $(X_0^*, Z_n^*(0))$ have limiting distribution (X_0^*, Z^*), and let (i, y) be a continuity point. For any $\epsilon > 0$, select a $\Gamma > 0$ where $\lim\limits_{t \to \infty} P(|Z_{N(t)} - k| > \Gamma) < \epsilon$, for all k. We know from theorem 2.1 there is an N where for all $n > N$,

$$P(\sup_{|x| \leq B} |Z_n^*(x) - Z_n^*(0)| > \epsilon) < \epsilon. \tag{2.2}$$

Select a $k > N$ where

$$|P(X_0^* = i, Z_k^*(0) \leq y + \epsilon) - P(X_0^* = i, Z^* \leq y + \epsilon)| < \epsilon. \tag{2.3}$$

From the i.i.d. nature of $\{U_n(i)\}$, we can see that for $g^{(n)}(x; i_1, \ldots, i_n)$ as defined in (1.3) that for all k

$$Z_{N(t)} \stackrel{d}{=} g^{(k)}(Z_{N(t) - k}; X_{N(t) - k + 1}, \ldots, X_{N(t)}).$$

Therefore

$$P(X_{N(t)} = i, Z_{N(t)} \leq y) = P(X_{N(t)} = i, g^{(k)}(Z_{N(t) - k}; Z_{N(t) - k + 1}, \ldots, X_{N(t)}) \leq y)$$

$$\leq P\{X_{N(t)} = i, g^{(k)}(Z_{N(t) - k}; X_{N(t) - k + 1}, \ldots, X_{N(t)}) \leq y, |Z_{N(t) - k}| \leq B,$$

$$\sup_{|x| \leq B} |g^{(k)}(x; X_{N(t) - k + 1}, \ldots, X_{N(t)}) - g^{(k)}(0; X_{N(t) - k + 1}, \ldots, X_{N(t)})| \leq \epsilon\}$$

$$+ P(|Z_{N(t) - k}| > \Gamma)$$

$$+ P(\sup_{|x| \leq B} |g^{(k)}(x; X_{N(t) - k + 1}, \ldots, X_{N(t)}) - g^{(k)}(0; X_{N(t) - k + 1}, \ldots, X_{N(t)})| > \epsilon).$$

It is therefore clear that

$$P(X_{N(t)} = i, \ Z_{N(t)} \leq y) \leq P(X_{N(t)} = i, \ g^{(k)}(0; \ X_{N(t)-k+1}, \ldots, X_{N(t)}) \leq y + \epsilon) \quad (2.4)$$

$$+ P(|Z_{N(t)-k}| > B)$$

$$+ P(\sup_{|x| \leq B} |g^{(k)}(x; \ X_{N(t)-k+1}, \ldots, X_{N(t)}) - g^{(k)}(0; \ X_{N(t)-k+1}, \ldots, X_{N(t)})| > \epsilon).$$

We first note that $\overline{\lim_{t \to \infty}} P(|Z_{N(t)-k}| > B) < \epsilon.$

Also, since the possible values of $X_{N(t)-k+1}, \ldots, X_{N(t)}$ are countable, and

for all $i_1, \ i_2, \ldots, \ i_k \in J$,

$$P(X_{N(t)} = i, \ g^{(k)}(0; \ X_{N(t)-k+1}, \ldots, X_{N(t)}) \leq y + \epsilon \ \Big| \ X_{N(t)-k+1} = i_1, \ldots, X_{N(t)} = i_k)$$

is a constant bounded by 1, we have

$$\lim_{t \to \infty} P(X_{N(t)} = i, \ g^{(k)}(0; \ X_{N(t)-k+1}, \ldots, X_{N(t)}) \leq y + \epsilon)$$

$$= P(X_0^* = i, \ g^{(k)}(0; \ X_{k-1}^*, \ldots, X_0^*) \leq y + \epsilon).$$

From a similar argument we have from (2.2) that

$$\lim_{t \to \infty} P(\sup_{|x| \leq B} |g^{(k)}(x; \ X_{N(t)-k+1}, \ldots, X_{N(t)}) - g^{(k)}(0; \ X_{N(t)-k+1}, \ldots, X_{N(t)})| > \epsilon)$$

$$\leq \epsilon.$$

Thus, combining (2.3) and (2.4) we have that

$$\varlimsup_{t\to\infty} P(X_{N(t)} = i, \ Z_{N(t)} \leq y) \leq P(X_0^* = i, \ Z^* \leq y + \epsilon) + 2\epsilon.$$

By similar arguments it can be established that

$$P(X_0^* = i, \ Z^* \leq y - \epsilon) - 2\epsilon \leq \varliminf_{t\to\infty} P(X_{N(t)} = i, \ Z_{N(t)} \leq y).$$

which completes the proof. $\square$

Of course, the crucial condition in the application of theorem 2.3 to establish that if $\mu < \infty$ then (X_n, Z_n) converging implies $(X(t), Z(t))$ converges is the condition that for all k

$$\varlimsup_{t\to\infty} P(|Z_{N(t)} - k| > L) < \epsilon,$$

which implies that a bound can be selected that will apply for any k. We now establish that for semi-Markov process on an arbitrary state space with a regeneration point with finite expected return time, that such uniform bounds exist.

Let $\{Y_n, T_n\}$ be a well-defined Markov renewal process on some normed state space S (for a more detailed definition, see Cinlar (1969)).

As usual, we define $Y(t) = Y_{N(t)}$ and for any $y \in S$, random variable Z and event A, we define

$$E_y(Z) = E(Z|Y_0 = y), \text{ and } P_y(A) = P(A|Y_0 = y).$$

We then assume that there exists a point $x_0 \in S$ where for all $y \in S$,

$$P_y(\bigcup_{n=1}^{\infty}(Y_n = x_0)) > 0, \qquad (2.5)$$

and for $N = \inf \{n: Y_n = x_0\}$ and $T = T_N$,

$$E_{x_0}(N) < \infty, \qquad (2.6)$$

$$E_{x_0}(T) < \infty. \qquad (2.7)$$

As is shown in Çinlar (1975), such regeneration points make the behavior of semi-Markov processes quite tractable. In particular, we have the following theorem.

THEOREM 2.4. <u>Let (Y_n, T_n) be a Markov renewal process on a normed space S where there exists an $x_0 \in S$ satisfying (2.5), (2.6) and (2.7). Then for all $\epsilon > 0$, there is a $\Gamma > 0$ where for all $k > 0$</u>

$$\overline{\lim_{t \to \infty}} \ P(||Y_{N(t)} - k|| > k) < \epsilon.$$

PROOF. As is shown in Orey (1971), we have that $P_y(Y_n = x_0 \text{ i.o.}) = 1$ for almost all y, and there exists a stationary probability measure $\mu(\cdot)$ where for all $A \subset S$,

$$\mu(A) = \int_S \mu(dx) P(Y_1 \in A | Y_0 = x). \qquad (2.8)$$

Therefore, for almost all $y \in S$,

$$\lim_{t \to \infty} P_y(||Y_{N(t)} - k|| > B) = \lim_{t \to \infty} P_{x_0}(||Y_{N(t)} - k|| > B),$$

and since $\lim_{t \to \infty} N(t) = \omega$ a.s., we need only consider

$$\lim_{t \to \infty} P_{x_0}(||Y_{N(t)} - k|| > B, N(t) \geq k).$$

Clearly x_0 is a regeneration point, and therefore

$$P_{x_0}(||Y_{N(t)} - k|| > B, N(t) \geq k) = P_{x_0}(||Y_{N(t)} - k|| > B, N(t) \geq k, T > t)$$

$$+ P_{x_0}(||Y_{N(t)} - k|| > B, N(t) \geq k, 0 \leq N(t) - N(T) < k)$$

$$+ \int_0^t P_{x_0}(||Y_{N(t)} - k|| > B, N(t) \geq k, N(t) - N(s) \geq k | T = s) dP_{x_0}(T \leq s). \quad (2.9)$$

Since T is a time of regeneration, we have,

$$\int_0^t P_{x_0}(||Y_{N(t)} - k|| > B, N(t) \geq k, N(t) - N(s) \geq k) | T = s) dP_{x_0}(T \leq s)$$

$$= \int_0^t P_{x_0}(||Y_{N(t - s)} - k|| > B, N(t - s) \geq k) dP_{x_0}(T \leq s).$$

Therefore we have that (2.9) is a renewal equation, and if it can be shown that

$$P_{x_0}(||Y_{N(t)} - k|| > B, \; N(t) \ge k, \; T > t)$$

$$+ P_{x_0}(||Y_{N(t)} - k|| > B, \; N(t) \ge k, \; 0 \le N(t) - N(T) < k) \qquad (2.10)$$

is directly Riemann integrable, then the basic renewal theorem will

yield $\lim_{t \to \infty} P_{x_0}(||Y_{N(t)} - k|| > B, \; N(t) \ge k)$.

Clearly, the first term of (2.10) is directly Riemann integrable, since

$$P_{x_0}(||Y_{N(t)} - k|| > B, \; N(t) \ge k, \; T > t) \le P_{x_0}(T > t)$$

which is nonincreasing with $\int_0^\infty P_{x_0}(T > t) dt = E_{x_0} T < \infty$.

Also $P_{x_0}(||Y_{N(t)} - k|| > B, \; N(t) \ge k, \; 0 \le N(t) - N(T) < k)$

$$\le P_{x_0}(T > t) + P_{x_0}(N(t) - N(T) < k, \; T \le t)$$

$$= 1 - P_{x_0}(N(t) - N(T) \ge k, \; T \le t).$$

Because $P_{x_0}(N(t) - N(T) \ge k, \; T \le t)$ is nondecreasing, if we can show

$\int_0^\infty [P_{x_0}(T > t) + P_{x_0}(N(t) - N(T) < k, \; T \le t)] dt < \infty$, we will have established

the directly Riemann integrability of the second term of (2.10). First

note that $\int_0^\infty P_{x_0}(T > t) dt < \infty$.

Also

$$\int_0^\infty P_{x_0}(N(t) - N(T) < k, \ T \leq t)dt$$

$$= \int_0^\infty \int_0^t P_{x_0}(N(t-s) < k)dP_{x_0}(T \leq s)dt$$

$$= \int_0^\infty (\int_s^\infty P_{x_0}(N(t-s) < k)dt)dP_{x_0}(T \leq s)$$

$$= \int_0^\infty (\int_0^\infty P_{x_0}(T_k > t)dt)dP_{x_0}(T \leq s) = E_{x_0}T_k.$$

Since $E_{x_0}T < \infty$, it clearly follows that $E_{x_0}T_k < kE_{x_0}T < \infty$, which establishes the direct Riemann integrability.

Therefore, from the basic renewal theorem, we have that

$$\lim_{t \to \infty} P_{x_0}(||Y_{N(t)} - k|| > B)$$

$$= (E_{x_0}T)^{-1}[\int_0^\infty P_{x_0}(||Y_{N(t)} - k|| > B, \ N(t) \geq k, \ T > t)dt$$

$$+ \int_0^\infty P_{x_0}(||Y_{N(t)} - k|| > B, \ N(t) \geq k, \ 0 \leq N(t) - N(T) < k)dt]. \qquad (2.11)$$

To establish the uniformity of the bound B, let us define

$$Y^* = \sup_{0 \leq t \leq T} (||Y_{N(t)}||).$$

Since $E_{x_0}T < \infty$, and $Y^* < \infty$ a.s., we have from the dominated convergence theorem that $\lim_{B \to \infty} \int_0^\infty P_{x_0}(Y^* > B, \ T > t)dt = 0$.

Therefore, for all $\epsilon > 0$, there is a $B > 0$ where for all $k > 0$,

$$\int_0^\infty P_{x_0}(||Y_{N(t)} - k|| > B, N(t) \geq k, T > t)dt \leq \int_0^\infty P_{x_0}(Y^* > B, T > t)dt < \epsilon. \quad (2.12)$$

Also,

$$\int_0^\infty P_{x_0}(||Y_{N(t)} - k|| > B, N(t) \geq k, 0 \leq N(t) - N(T) < k)dt$$

$$\leq \int_0^\infty P_{x_0}(Y^* > B, N(t) \geq k, N(t) - N(T) < k, T \leq t)dt$$

$$= \int_0^\infty \int_0^t P_{x_0}(Y^* > B, N(t) \geq k, N(t) - N(s) < k | T = s)dP_{x_0}(T \leq s)dt. \quad (2.13)$$

From (2.13) it follows that

$$\int_0^\infty P_{x_0}(||Y_{N(t)} - k|| > B, N(t) \geq k, 0 \leq N(t) - N(T) < k)dt$$

$$\leq \int_0^\infty \int_0^t \sum_{j=0}^{k-1} P_{x_0}(Y^* > B, N(s) \geq k - j, N(t) - N(s) = j | T = s)dP_{x_0}(T \leq s)dt$$

$$= \int_0^\infty \int_s^\infty \sum_{j=0}^{k-1} P_{x_0}(Y^* > B, N(s) \geq k - j | T = s)P_{x_0}(N(t - s) = j)dt \, dP_{x_0}(T \leq s)$$

$$= \int_0^\infty \sum_{j=0}^{k-1} P_{x_0}(Y^* > B, N(s) \geq k - j | T = s)\lfloor \int_0^\infty P_{x_0}(N(t) = j)dt \rfloor dP_{x_0}(T \leq s). \quad (2.14)$$

It is easily seen that

$$\int_0^\infty P_{x_0}(N(t) = j)dt = \int_0^\infty [P_{x_0}(T_{j+1} > t) - P_{x_0}(T_j > t)]dt$$

$$= E_{x_0} T_{j+1} - E_{x_0} T_j = E_{x_0} (T_{j+1} - T_j) = \int_S P^j(x_0, dy) E_y T_1,$$

where $P^n(x_0, A) = P(Y_n \in A | Y_0 = x)$. From (2.8) it follows that

$$\int_S \mu(dx) \int_S P^n(x, dy) E_y T_1 = \int_S \mu(dx) E_y T_1 = \mu(x_0) E_{x_0} T \text{ (the last equality can}$$

be found in Çinlar (1975)).

Therefore, $\mu(x_0) E_{x_0}(T_{j+1} - T_j) \leq \mu(x_0) E_{x_0} T$, and we once again find that

$$E_{x_0}(T_{j+1} - T_j) \leq E_{x_0} T.$$

Therefore, from (2.14) we have that

$$\int_0^\infty P_{x_0}(||Y_{N(t)} - k|| > B, \ N(t) \geq k, \ 0 \leq N(t) - N(T) < k) dt$$

$$\leq (E_{x_0} T) \sum_{j=1}^k \int_0^\infty P_{x_0}(Y^* > B, \ N(s) \geq j | T = s) dP_{x_0}(T \leq s)$$

$$= (E_{x_0} T) \sum_{j=1}^k P_{x_0}(Y^* > B, \ N(T) \geq j)$$

$$\leq (E_{x_0} T) \sum_{j=1}^\infty P_{x_0}(N(T) \geq j) = (E_{x_0} T) E_{x_0} N < \infty. \tag{2.15}$$

Again by the dominated convergence theorem, for all $\epsilon > 0$, there is a $B > 0$ where

$$(E_{x_0} T) \sum_{j=1}^\infty P_{x_0}(Y^* > B, \ N(T) \geq j) < \epsilon.$$

From (2.15) it then follows that for all $k > 0$,

$$\int_0^\infty P_{x_0}(||Y_{N(t)} - k|| > B, \; N(t) \geq k, \; 0 \leq N(t) - N(T) < k)dt < \epsilon,$$

and therefore (2.11) coupled with (2.12) and (2.15) completes the proof. □

Of course, it must be pointed out that should $\{(X_n, Z_n)\}$ have a point of regeneration the convergence of $\{X(t), Z(t)\}$ follows directly from the basic renewal theorem, and theorem 2.3 is unnecessary. This situation is not the primary situation of interest. But even in this case, as will be shown, the theorem can lead to a more satisfying answer to the limit behavior of $\{X(t), Z(t)\}$ than the generally intractable integral answer which results from the basic renewal theorem.

The primary situation of interest is when $\{X_n, Z_n\}$ has no point of regeneration, but has instead the following two properties

1) $Z_n(x) \geq 0$ a.s., $\forall n, \forall x,$

(2.16)

2) if $x \geq y$, then $Z_n(x) \geq Z_n(y)$ a.s..

In this case, we will show that should $\{X_n, Z_n\}$ converge in distribution, the uniform boundedness condition is satisfied by bounding $\{X_n, Z_n\}$ by another chain $\{X_n, W_n\}$ with a regeneration point.

THEOREM 2.5.   If $\beta < \infty$, $\{X_n, Z_n\} \to \{X, Z\}$, and if property (2.16) is valid, then

$$\lim_{t \to \infty} P(X(t) = i, \; Z(t) \leq y) = \lim_{n \to \infty} \{X_0^* = i, \; Z_n^*(0) \leq y\},$$

for all continuity points y.

PROOF. If there is no regeneration point, then from theorem 1.1 we know that for some i_0, and any $x > 0$, there is an $A > 0$, a $B > A$, an $\epsilon > 0$, and an m where

$$A = \sup\{D: \lim_{n \to \infty} P(X_n = i_0, Z_n(x) \leq D) = 0\},$$

(2.17)

$$P(X_m = i_0, Z_m(A) > B, X_{m-1} \neq i_0, \ldots, X_1 \neq i_0 \mid X_0 = i_0) > \epsilon.$$

Assume without loss of generality that $A = 0$. We can then define two new Markov chains $\{(X_n, W_n)\}$ and $\{X_n, V_n\}$, where the transition probabilities for $\{X_n, W_n\}$ are given for arbitrary set C by

$$P(X_n = i, W_n \in C \mid X_{n-1} = j, W_{n-1} = y)$$

$$= \begin{cases} P(X_n = i, Z_n \in C \mid X_{n-1} = j, Z_{n-1} = y) & \text{if } i \neq i_0 \text{ or } C \cap [0, B] = \emptyset \\ P(X_n = i_0, Z_n \in C \cup [0, B] \mid X_{n-1} = j, Z_{n-1} = y) & \text{if } i = i_0, B \in C \quad (2.18) \\ 0 & \text{if } i = i_0, C \subset [0, B), \end{cases}$$

and those for $\{X_n, V_n\}$ are given by

$$P(X_n = i, V_n \in C \mid X_{n-1} = j, W_{n-1} = v)$$

$$= \begin{cases} P(X_{n-1}, Z_n \in C \mid X_{n-1} = j, Z_{n-1} = y) & \text{if } i \neq i_0 \text{ or } C \cap [0, B] = \emptyset \\ P(X_n = i_0, Z_n \in C \cup [0, B] \mid X_{n-1} = j, Z_{n-1} = y) & \text{if } i = i_0, 0 \in C \quad (2.19) \\ 0 & \text{if } i = i_0, C \subset (0, B]. \end{cases}$$

From the above definitions, we can see that W_n and V_n are merely versions of Z_n in which whenever $X_n = i_0$ and $Z_n \le B$, the value of Z_n is immediately changed to B and 0 respectively, and then the process is restarted.

Clearly, from (2.16) it follows that

$$V_n(x) \le Z_n(x) \le W_n(x) \text{ a.s.}.$$

Let us define

$$N_z = \inf \{n > 1: X_n = i_0, \ Z_n \le B\},$$

$$N_w = \inf \{n > 1: X_n = i_0, \ W_n \le B\},$$

$$N_v = \inf \{n > 1: X_n = i_0, \ V_n \le B\}.$$

From Cinlar (1975), we need only show that $E_{(i_0, B)} N_w < \infty$ to have $\{X_n, W_n\}$ satisfying theorem 2.4. The other condition follows, since $E_{(i_0, B)} N_w < \infty$ implies there is a stationary probability measure μ with $\mu(i_0, B) > 0$, in which case for $T_w = T_{N_w}$,

$$E_{(i_0, B)} T_w = [\mu(i_0, B)]^{-1} \sum_{i \in J} \int_0^\infty \mu(i, dy) E_{(i, y)} T_1 = [\mu(i_0, B)]^{-1} \beta < \infty.$$

Should $\{X_n, Z_n\}$ be ergodic this would follow immediately, since $E_{(i_0, B)} N_w = E_{(i_0, B)} N_z$. Unfortunately without a ϕ-irreducibility condition (for definition, see Orey (1971)), the convergence of $\{X_n, Z_n\}$ is not sufficient for ergodicity. However, it is clear from (2.16) that for any $x < y$, and for $x > B$ that

$$E_{(i_0, x)} N_w \geq E_{(i_0, y)} N_w,$$

$$(2.20)$$

$$E_{(i_0, x)} N_w = E_{(i_0, x)} N_v.$$

Thus, it is sufficient to show that for some $x > B$, $E_{(i_0, x)} N_v < \infty$.

Let us first establish that $\{X_n, V_n\}$ is ergodic. Since $(i_0, 0)$ is a regeneration point, it follows from Tweedie (1975) that if $\lim_{n \to \infty} P(X_n = i_0, V_n = 0 | X_0 = i, V_0 = z) > 0$ for some (i, z), then $\{X_n, V_n\}$ is ergodic. From (2.16) and (2.19) it is clear that

$$P(X_n = i_0, V_n = 0 | X_0 = i, V_0 = z) \geq P(X_n = i_0, Z_n \leq B | X_0 = i, Z_0 = z).$$

Therefore, we have $\{X_n, V_n\}$ is ergodic since for all (i, z),

$$\lim_{n \to \infty} P(X_n = i_0, Z_n \leq B | X_0 = i, Z_0 = z) > \epsilon.$$

From the ergodicity of $\{X_n, V_n\}$, it follows that $E_{(i_0, 0)} N_v < \infty$.

It can be shown that

$$E_{(i_0, 0)} N_v = \sum_{i=1}^{m} i P_{(i_0, 0)} (N_v = i)$$

$$+ P_{(i_0, 0)} (N_v > m) \sum_{j \in J} \int_0^\infty P_{(i_0, 0)} (j, dx | N_v > m)(m + E_{(j, x)} N_v).$$

From (2.17) it follows that $E_{(i_0, 0)} N_v < \infty$ implies that $E_{(i_0, x)} N_v < \infty$ for almost all x in the non-empty support of $P_{i_0, 0}^{(m)} (i_0, (B, \infty) | N_v > m)$,

Which in turn implies that $E_{(i_0, B)} N_w < \infty$ from (2.20).

Since $E_{(i_0, B)} N_w < \infty$, we have that theorem 2.4 applies to $\{X_n, W_n\}$.

Therefore, for any $\varepsilon > 0$ there is a $B > 0$ where for all $k > 0$

$$\overline{\lim_{t \to \infty}} P(|W_{N(t)} - k| > B) < \varepsilon.$$

Since $Z_n(x) \leq W_n(x)$ a.s., this in turn implies

$$\overline{\lim_{t \to \infty}} P(|Z_{N(t)} - k| > B) \leq \overline{\lim_{t \to \infty}} P(|W_{N(t)} - k| > B) < \varepsilon,$$

which from theorem 2.3 completes the proof. $\square$

In the next section, we will apply these results to several storage models to determine the convergence behavior in continuous time.

3. APPLICATIONS

To illustrate the applications of the theorems in the previous section we will examine the behavior of several storage models. The following notation will be necessary. For any random variable Y define $E_\pi(Y)$ by

$$E_\pi(Y) = \sum_{j \in J} \pi_j \, E(Y | X_0 = j)$$

where $\underset{\sim}{\pi}$ is the stationary measure of the chain $\{X_n\}$.

EXAMPLE 1.

Let $\{U_n(i), n = 0, 1, \ldots\}$ be an i.i.d. sequence of real-valued random variables, independent of $\{X_n\}$ and of $\{U_n(j)\}$, for $j \neq i$.

Let $E_\pi|U_1| < \infty$, and let $E_\pi U_1 < 0$. We then define our contents in storage $Z_n(x)$ recursively by

$$Z_n(x) = \max(0, Z_{n-1}(x) + U_n(X_n)),$$

and $\qquad Z_0(x) \equiv x.$

Of course,

$$Z(t) = Z_{K(t)}(x).$$

This model is based on an early model for dam theory proposed by Moran (1954), which has shown itself to have diverse application (for example, in waiting times for queueing theory). In the present form, it has been examined by Balagopal (1979), Puri (1978), Senturia and Puri (1973, 1974), Puri and Woolford (1981), and Puri and Tollar (1985). In its most general form, Puri and Woolford (1981) had shown it converged in distribution when appropriately normalized when $E_\pi U_1 \geq 0$, and hypothesized it should converge without normalization when $E_\pi U_1 < 0$. This was shown to be true by Puri and Tollar (1985), who illustrated that $\{X_n, Z_n\}$ must have a renewal point $(i_0, 0)$, in which case renewal theory directly yields for $T = \inf \{t > T_1 : X(t) = i_0, Z(t) = 0\}$,

$$\lim_{t\to\infty} P(X(t) = i, Z(t) \in A) =$$

$$= [E_{(i_0, 0)}T]^{-1} \int_0^\infty P_{(i_0, 0)}(X(t) = i, Z(t) \in A, T > t) dP_{(i_0, 0)}(T \leq t).$$

$$(3.1)$$

Noting that

$$Z_n(x) = \max \left(\max_{1 \leq j \leq n} \left(\sum_{i=j+1}^{n} U_i(X_i) \right), \; x + \sum_{i=1}^{n} U_i(X_i) \right), \tag{3.2}$$

we see that

$$\hat{Z}_n(0) = \max_{-1 \leq j \leq n-1} \left(\sum_{i=0}^{j} U_i(\hat{X}_i) \right).$$

Since $E_\pi U_1 < 0$, we have from Chung (1967) that $\sum_{i=0}^{n} U_i(\hat{X}_i) \to -\infty$, a.s., in

which case

$$(\hat{X}_0, \; \hat{Z}_n) \longrightarrow (\hat{X}_0, \; \sup_{-1 \leq j} \left(\sum_{i=0}^{j} U_i(\hat{X}_i) \right)) \text{ a.s.}.$$

From theorem 1.1 this implies $(X_n, \; Z_n)$ converges in distribution if

we also observe from (3.2) that

$$\sup_{|x| \leq B} |Z_n(x) - Z_n(0)| \leq \max(0, \; B + \sum_{i=1}^{n} U_i(X_i)) \to 0 \text{ a.s.}.$$

Note that $Z_n(x) \geq 0$ a.s., and if $x \geq y$, then $Z_n(x) \geq Z_n(y)$ a.s.. As

such, we have the following theorem as a consequence of theorem 2.5.

THEOREM 3.1.   If $\beta < \infty$, $E_\pi U_1 < 0$, then

$$\lim_{t \to \infty} P(X(t) = i, \; Z(t) \leq y) = P(X_0^* = i, \; \sup_{-1 \leq j} \left(\sum_{i=0}^{j} U_i(X_i^*) \right) \leq y)$$

for all continuity points of y.

It should be observed that this form is substantially more satis-

fying than the integral form of (3.1) obtained by Puri and Tollar (1985).

While this by no means allows us to actually compute the distribution of the limit, Puri (1978) has found the Laplace transform of

$$\sup_{0 \leq j} \left(\sum_{i=1}^{j} U_i(X_i^*) \right)$$ for a two-state Markov chain $\{X_n\}$, which is a least a step in characterizing the limit distribution.

EXAMPLE II.

Let $\{(U_n(i), V_n(i)), n = 0, 1, \ldots\}$ be an i.i.d. sequence of bivariate real-valued random variables, independent of $\{X_n\}$ and of $\{U_n(j), V_n(j)\}$ for $j \neq i$.

We then define a proportional allocation scheme for our contents in storage, $Z_n(x)$, recursively by

$$Z_n(x) = U_n(X_n) \cdot Z_{n-1}(x) + V_n(X_n), \qquad (3.3)$$

and $Z_0(x) \equiv x$,

and again $Z(t) = Z_{N(t)}(x)$.

While this model has received no attention in the general framework, it has proven to be of interest in the simpler case where we define

$$Z_n(x) = U_n \cdot Z_{n-1}(x) + W_n,$$

for $\{(U_n, W_n)\}$ an i.i.d. sequence (see, for example, Barnard , Schenton, and Uppuluri (1967), Paulson and Uppuluri (1972), Vervaat (1979)).

Of particular interest is the observation that the techniques for
semi-Markov processes on arbitrary state spaces of Cinlar (1969), Athreya,
McDonald and Ney (1978a,b) and others have little chance for success on
this model, since there is no guarantee that the Markov chain $\{X_n, Z_n(x)\}$
is ϕ-irreducible. As such, while one can show the Markov chain converges
by a reversal argument, and therefore apply theorem 2.5, ergodicity of
$\{X_n, Z_n(x)\}$ cannot be demonstrated, so the typical starting assumption of
semi-Markov processes is missing.

We first establish a theorem on the convergence of $\{X_n, Z_n(x)\}$.

THEOREM 3.2.　　If $E_\pi \ell n|U_1| < 0$, $E_\pi(\ell n|V_1|)^+ < \infty$, then

$\lim_{n\to\infty} P(X_n = i, Z_n(x) \le y) = P(\hat{X}_0 = i, \hat{Z} \le y)$, for all continuity points y where

$$\hat{Z} = \sum_{i=0}^{\infty} V_i(\hat{X}_i) \prod_{j=0}^{i-1} U_i(\hat{X}_i) < \infty, \text{ a.s.}.$$

PROOF. From (3.3) it is easily seen that

$$Z_n(x) = x \prod_{i=1}^{n} U_i(X_i) + \sum_{i=1}^{n} V_i(X_i) \prod_{j=i+1}^{n} U_j(X_j),$$

so

$$\hat{Z}_n(x) = x \prod_{i=0}^{n-1} U_i(\hat{X}_i) + \sum_{i=0}^{n-1} V_i(\hat{X}_i) \prod_{j=0}^{i-1} U_j(\hat{X}_j).$$

Since $\ell n| \prod_{i=0}^{k} U_i(X_i)| = \sum_{i=0}^{k} \ell n|U_i(X_i)|$, we have from Chung (1967) that

$E_\pi \ell n|U_1| < 0$, implies

$$\lim_{n\to\infty} \frac{1}{n} \ell n |\prod_{i=0}^{n} U_i(X_i)| < 0, \quad a.s.. \qquad (3.4)$$

Therefore,

$$\sup_{|x|\leq B} |Z_n(x) - Z_n(0)| = B \prod_{i=1}^{n} |U_i(X_i)| \to 0 \quad a.s.,$$

so condition (1.4) is satisfied. Therefore, from theorem 1.1 we need only examine $\hat{Z}_n(0)$.

From (3.4) we have for all $\epsilon > 0$, there is a δ, $0 < \delta < 1$, and an N where for $n > N$,

$$P(\ell n |\prod_{i=0}^{n-1} U_i(\hat{X}_i)| > n\ell n\delta, \quad \forall \; n > N) < \epsilon,$$

so

$$P(|\prod_{i=0}^{n-1} U_i(\hat{X}_i)| > \delta^n, \quad \forall \; n > N) < \epsilon. \qquad (3.5)$$

Therefore,

$$P(|\hat{Z}_m(0) - \hat{Z}_n(0)| > \epsilon, \quad \forall \; m > n > N)$$

$$\leq P(\sum_{i=n}^{\infty} |V_i(\hat{X}_i)| \cdot |\prod_{j=0}^{i-1} U_j(\hat{X}_j)| > \epsilon)$$

$$\leq P(\sum_{i=0}^{\infty} |V_i(X_i)| \delta^{n+i} > \epsilon) + \epsilon, \qquad (3.6)$$

where the last inequality follows from (3.5) and from $\{\hat{X}_n\}$ having the stationary distribution.

For any $x > 1$,

$$\sum_{i=0}^{\infty} P(|V_i(\hat{X}_i)| > x^i) = \sum_{j \in J} \pi_j \sum_{i=0}^{\infty} P(|V_i(j)| > x^i)$$

$$= \sum_{j \in J} \pi_j \sum_{i=0}^{\infty} P(\ell n |V_i(j)| > i \ell n x)$$

$$\leq \sum_{j \in J} \pi_j (\ell n x)^{-1} (1 + E(\ell n |V_1(j)|)^+)$$

$$= (\ell n x)^{-1} (1 + E_\pi (\ell n |V_1|)^+) < \infty.$$

As such, by the Borel-Cantelli lemma, for any $x > 1$

$$P(|V_i(\hat{X}_i)| > x^i \ \text{i.o.}) = 0.$$

Therefore, for any $x < \delta^{-1}$, since $\sum_{i=0}^{\infty} x^i \delta^i < \infty$,

$$|\sum_{i=0}^{\infty} |V_i(\hat{X}_i)| \delta^i - \sum_{i=0}^{\infty} x^i \delta^i| < \infty, \ \text{a.s.},$$

which implies that $\sum_{i=0}^{\infty} |V_i(X_i)| \delta^i < \infty$, a.s..

As such, we find from (3.6) that for N sufficiently large,

$$P(|\hat{Z}_m(0) - \hat{Z}_n(0)| > \epsilon, \forall m > n > N) < \epsilon,$$

and we have that $\hat{Z}_n(0)$ converges almost surely to some random variable Z, which completes the proof. $\Box$

To complete the analysis of $Z(t)$, we once again appeal to the results of the previous section.

THEOREM 3.3.   If $\beta < \infty$, $E_\pi \ell n |U_1| < 0$, $E_\pi (\ell n |V_1|)^+ < \infty$, then

$$\lim_{t \to \infty} P(X(t) = i, \ Z(t) \leq y) = P(X_0^* = i, \ Z^* \leq y), \ \text{for all continuity points } y, \text{ where}$$

$$Z^* = \sum_{i=0}^{\infty} V_i(X_i^*) \prod_{j=0}^{i-1} U_i(X_i^*) < \infty, \ a.s..$$

PROOF.   While theorem 2.5 cannot be appealed to directly, clearly

$$|Z_n(x)| \leq W_n(|x|) \equiv |x| \prod_{i=1}^{n} |U_i(X_i)| + \sum_{i=1}^{n} |V_i(X_i)| \prod_{j=i+1}^{n} |U_j(X_j)|,$$

where $\{X_n, W_n(|x|)\}$ is also a Markov chain.   Since $E_\pi \ell n |U_1| < 0$,

$E_\pi (\ell n |V_1|)^+ < \infty$, theorem 3.2 yields that $(X_n, W_n(|x|) \longrightarrow (X, W)$ in

distribution.

Also, property (2.16) is clearly satisfied for $W_n(|x|)$.   Therefore, the

proof of theorem 2.5 is sufficient to establish that for any $\epsilon > 0$, and

all $k > 0$, there exists a B where

$$\overline{\lim_{t \to \infty}} \ P(|Z_n(t) - k| > B) \leq \overline{\lim_{t \to \infty}} \ P(|W_n(t) - k| > B) < \epsilon,$$

and therefore theorem 2.3 can be applied to complete the proof.   □

4. CONCLUSION

In most cases when a storage model is defined on a Markov renewal process, the convergence of even the discrete time version cannot be determined from general Markov chain arguments. As such, some technique like reversing the process must be used. Therefore, if the conditions in section 2 are valid, we essentially get the convergence of the continuous time process for free. Of course, it is clear that conditions such as (1.4) and (2.16) can be extended directly to multiple dimensions, and the results will remain valid. However, for more general spaces, the counterpart to (2.16) is not readily apparent to us at present.

Crucial to the usefulness of the results are conditions to guarantee the uniform boundedness of $Z_{N(t) - k}$. The technique of bounding $(X_n, Z_n(x))$ by another Markov chain with a regeneration point seems very powerful. The conditions given can clearly by generalized. For example, if it can be shown for a measure π that π is invariant with respect to $\{X_n, Z_n(x)\}$, then we need not have $Z_n(x) \geq 0$ in (2.16) for the results to still be true. It should be noted that it seems that $Z_n(x) \geq 0$ should be unnecessary in (2.16) even under the presented conditions. However, we have been unable to show that this is the case.

Certainly there must be methods other than the bounding arguments used which could be considered in establishing the uniform boundedness of $Z_{N(t) - k}$. Such conditions would be an area of major interest in the applicability of the present work. And of course, perhaps uniform boundedness is not the only condition which will let a theorem like theorem 2.3 be valid.

Of major interest would be to eliminate the underlying denumerable state semi-Markov process that was so essential to the present work, and instead specify a reversing technique for semi-Markov processes on arbitrary state spaces with no such structure. However, even if we can specify a way to reverse the Markov chain, we have no corresponding version of lemma 2.2 upon which to "build" the reversed process in continuous time. Therefore, at present, we see no hope of this technique being generalizable in this direction. Except in some very artificial scenarios, it seems that establishing the equivalent of lemma 2.2 is substantially more difficult than establishing the behavior of the process directly. However, it is certainly possible that techniques different than those used here could make the method applicable.

LIST OF REFERENCES

Athreya, K. B., McDonald, D., and Ney, P. (1978a). Coupling and the renewal theorem, Amer. Math. Monthly, 85, 809-814.

Athreya, K. B., McDonald, D., and Ney, P. (1978b). Limit theorems for semi-Markov processes and renewal theory for Markov chains, Ann. Probability, 6, 788-797.

Athreya, K. B., and Ney, P. (1978). Limit theorems for semi-Markov processes, Bull. Austral. Math. Soc., 19, 283-294.

Balagopal, K. (1979). Some limit theorems for the general semi-Markov storage model, J. Appl. Prob., 16, 607-617.

Barnard, S. R., Shenton, L. R., and Uppuluri, V. R. R. (1967). Stochastic models for the distribution of radioactive material in a connected system of compartments. Proc. Fifth Berkeley Symp. Math. Statist. Prob., 481-510.

Chung, K. L. (1967). Markov Chains with Stationary Transition Probabilities, Springer-Verlag, New York.

Cinlar, E. (1969). On semi-Markov processes on arbitrary spaces, Proc. Cambridge. Phil. Soc., 66, 381-392.

Cinlar, E. (1975). Introduction to Stochastic Processes, Prentice Hall, New Jersey.

Karlin, S. and Taylor, H. M. (1975). A First Course in Stochastic Processes, Academic Press, New York.

Kemeny, J. G., Snell, J. L., and Knapp, A. W. (1976). Denumerable Markov Chains, Springer-Verlag, New York.

Kesten, H. (1974). Renewal theory for Markov chains, Ann. Probability, 3, 355-387.

Moran, P. A. P. (1954). A probability theory of dams and storage systems, Aust. J. Appl. Sci., 5, 116-124.

Nummelin, E. (1978). A splitting technique for Harris recurrent Markov chains, Z. Wahrsch. Verw. Geb., 43, 309-318.

Orey, S. (1971). Limit Theorems for Markov Chain Transition Probabilities, Van Nostrand Reinhold, London.

Paulson, A. S. and Uppuluri, V. R. R. (1972). Limit laws on a sequence determined by a random difference equation governing a one-compartment system. Math. Biosci. 13, 325-333.

Puri, P. S. (1978). A generalization of a formula of Pollaczek and Spitzer as applied to a storage model, Sankhyā, A, 40, 237-252.

Puri, P. S. and Tollar, E. S. (1985). On the limit behavior of certain quantities in a subcritical storage model, Adv. Appl. Prob., 17, 443-459.

Puri, P. S. and Woolford, S. W. (1981). On a generalized storage model with moment assumptions, J. Appl. Prob., 18, 473-481.

Senturia, J. and Puri, P. S. (1973). A semi-Markov storage model, Adv. Appl. Prob., 5, 362-378.

Senturia, J. and Puri, P. S. (1974). Further aspects of a semi-Markov storage model, Sankhyā, A, 36, 369-378.

Tweedie, R. L. (1975). Sufficient conditions for ergodicity and recurrence of Markov chains on a general state space, Stoc. Proc. Appl., 3, 385-403.

Vervaat, W. (1979). On a stochastic difference equation and a representation of non-negative infinitely divisible random variables. Adv. Appl. Prob., 11, 750-783.

END

DTIC

7 - 86